QP Code: 22/PT/13/XA(i)

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-10A(i) : ADVANCED DIFFERENTIAL GEOMETRY

(Pure Mathematics)

(Spl. Paper)

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Notations have their usual meanings)

[A manifold always means a differentiable manifold of class c^{∞}]

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Define chart on a manifold.

   b) Using $[X, X] = \theta$, show that $[X, Y] = -[Y, X]$.

   c) Find (f_*) where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

   $f(x^1, x^2) = (x^1 e^{x^2} + x^2, x^1 e^{x^2} - x^2)$.

   d) When is a vector field on a manifold said to be complete ?

   e) Test whether ω is closed or not, where

   $\omega = xy\,dx + \left(\frac{1}{2}x^2 - y\right)dy$.

   f) Show that on a Lie group G

   $L_a L_b \neq L_b L_a$, $\forall a, b$ in G, when L_a is the left-translation on G.

   g) Define linear connection on a manifold in the sense of Koszul.

2. a) Define Atlas on a manifold.

   b) Prove that a circle in the xy plane of $\mathbb{R}^2$ is an 1-dimensional manifold. 3 + 7

3. a) Show that, for every smooth vector field *X, Y* on a manifold *M*, $[X, fY] = f[X, Y] + (Xf)Y$, f being smooth function on *M*.

   b) Find the integral curve of a null vector field on *M*. 6 + 4

4. a) Define *f*-related vector fields on a manifold. If X_i, Y_i $(i = 1, 2)$ are *f*-related vector fields on manifolds *M* and *N* respectively, show that $[X_1, X_2]$ and $[Y_1, Y_2]$ are also *f*-related.

   b) Show that a set of 1-forms $\{\omega_1, \omega_2, \ldots, \omega_k\}$ is linearly dependent if $\omega_1 \wedge \omega_2 \wedge \ldots\ldots \wedge \omega_k = 0$. (2 + 4) + 4

5. a) If ω_1, ω_2 are left invariant differential forms on a Lie group, show that $\omega_1 \wedge \omega_2$ is also so.

   b) State and prove Maurer-Cartan Equation. 3 + (2 + 5)

6. a) Define Torsion tensor field on a manifold and show that it is skew-symmetric.

   b) Define curvature tensor field *R* on a manifold *M*. Show that $R(fX, Y)z = fR(X, Y)Z$, *f* being c^{∞} function on *M* and *X, Y, Z* are all smooth vector fields on *M*. 3 + (2 + 5)

7. a) Define Riemannian Manifold. 2

   b) Let ∇ be a metric connection on a Riemannian Manifold (M,g) and $\tilde{\nabla}$ be another linear connection given by

   $$\tilde{\nabla}_X Y = \nabla_X Y + T(X, Y)$$

   where *T* is the torsion tensor of *M*. Show that the following conditions are equivalent

   (i) $\tilde{\nabla}_g = 0$ (ii) $g((T(X, Y), Z) + g(Y, T(X, Z)) = 0$ 4 + 4